Define the polarization state of a field as a 2D vector—
\textit{Jones vector}—containing the two complex amplitudes:

\[
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
\]

For many purposes, we only care about the relative values:

(alternatively normalize this vector to unity magnitude)

Specifically:

- $0^\circ$ linear (\(x\)) polarization: \(E_y/E_x = 0\)
- $90^\circ$ linear (\(y\)) polarization: \(E_y/E_x = \infty\)
- $45^\circ$ linear polarization: \(E_y/E_x = 1\)

Arbitrary linear polarization:

\[
\frac{E_y}{E_x} = \sin(\alpha) / \cos(\alpha) = \tan(\alpha)
\]

\[\begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
\]

The Mathematics of Polarization

Circular (or Helical) Polarization

\[
E_x(z,t) = E_0 \cos(kz - \omega t)
\]

\[
E_y(z,t) = E_0 \sin(kz - \omega t)
\]

Or, more generally,

\[
E_x(z,t) = \text{Re}\left[ E_0 \exp\left[ i(kz - \omega t) \right] \right]
\]

\[
E_y(z,t) = \text{Re}\left[ -i E_0 \exp\left[ i(kz - \omega t) \right] \right]
\]

Here, the complex amplitude of the \(y\)-component is \(-i\) times the complex amplitude of the \(x\)-component.

So the components are always $90^\circ$ out of phase.

Right vs. Left Circular (or Helical) Polarization

\[
E_x(z,t) = E_0 \cos(kz - \omega t)
\]

\[
E_y(z,t) = -E_0 \sin(kz - \omega t)
\]

Or, more generally,

\[
E_x(z,t) = \text{Re}\left[ E_0 \exp\left[ i(kz - \omega t) \right] \right]
\]

\[
E_y(z,t) = \text{Re}\left[ +i E_0 \exp\left[ i(kz - \omega t) \right] \right]
\]

Here, the complex amplitude of the \(y\)-component is \(+i\) times the complex amplitude of the \(x\)-component.

So the components are always $90^\circ$ out of phase, but in the other direction.

The resulting \(E\)-field rotates counterclockwise around the \(k\)-vector (looking along \(k\)).

The resulting \(E\)-field rotates clockwise around the \(k\)-vector (looking along \(k\)).
Unequal arbitrary-relative-phase components yield elliptical polarization

\[ E_x(z,t) = E_{0x} \cos(kz - \omega t) \]
\[ E_y(z,t) = E_{0y} \cos(kz - \omega t - \theta) \]

where \( E_{0x} \neq E_{0y} \).
Or, more generally,

\[ E_x(z,t) = \text{Re}\{E_{0x} \exp[i(kz - \omega t)]\} \]
\[ E_y(z,t) = \text{Re}\{E_{0y} \exp[i(kz - \omega t)]\} \]

where \( E_{0x} \) and \( E_{0y} \) are arbitrary complex amplitudes.

The resulting E-field can rotate clockwise or counter-clockwise around the k-vector (looking along k).

When the phases of the x- and y-polarizations fluctuate, we say the light is unpolarized.

\[ E_x(z,t) = \text{Re}\left\{A_x \exp[i(kz - \omega t - \theta_x(t))]\right\} \]
\[ E_y(z,t) = \text{Re}\left\{A_y \exp[i(kz - \omega t - \theta_y(t))]\right\} \]

where \( \theta_x(t) \) and \( \theta_y(t) \) are functions that vary on a time scale slower than \( 1/\omega \), but faster than you can measure.

The polarization state (Jones vector) will be:

\[ \left[ \frac{A_x}{A_y} \exp[i(\theta_y(t) - \theta_x(t))] \right] \]

In practice, the amplitudes vary, too!

As long as the time-varying relative phase, \( \theta_x(t) - \theta_y(t) \), fluctuates, the light will not remain in a single polarization state and hence is unpolarized.

Definitions

**Thickness of wave plates**

When a wave plate has less than \( 2\pi \) relative phase delay, we say it’s a zero-order wave plate. Unfortunately, they tend to be very thin.

Solve for \( d \) to find the thickness of a zero-order quarter-wave plate:

\[ \frac{2\pi}{\lambda} |n_o - n_e| d = \frac{\pi}{2} \]

\[ d = \frac{\lambda}{4|n_o - n_e|} \]

Using green light at 500 nm and quartz, whose refractive indices are \( n_e - n_o = 1.5534 - 1.5443 = 0.0091 \), we find:

\[ d = 13.7 \mu m \]

This is so thin that it is very fragile and very difficult to manufacture.
Depolarization by reflection or transmission

Suppose that 45° polarization is incident on an interface, which has different parallel (x) and perpendicular (y) reflection coefficients.

Incident light fields:
\[ E_x(z,t) = \text{Re}\left\{ E_0 \exp\left[i(kz - \omega t)\right]\right\} \]
\[ E_y(z,t) = \text{Re}\left\{ E_0 \exp\left[i(kz - \omega t)\right]\right\} \]

Reflected light fields:
\[ E_x(z,t) = \text{Re}\left\{ r_x E_0 \exp\left[i(kz - \omega t)\right]\right\} \]
\[ E_y(z,t) = \text{Re}\left\{ r_y E_0 \exp\left[i(kz - \omega t)\right]\right\} \]

Unless light is purely parallel or perpendicularly polarized (or incident at 0°), polarization rotation will occur in reflection (or transmission).

Fresnel Reflection and Depolarization

Fresnel reflections are a common cause of polarization rotation. The ratio of polarization-component strengths will change on transmission through or reflection off an interface. This effect is particularly strong near Brewster’s angle.

Fresnel Equations

We would like to compute the fraction of a light wave reflected and transmitted by a flat interface between two media with different refractive indices.

\[ \frac{E_{0r}}{E_{0i}} \quad \text{for the perpendicular polarization} \]
\[ \frac{E_{0t}}{E_{0i}} \quad \text{for the parallel polarization} \]

where \( E_{0r}, E_{0t}, \) and \( E_{0i} \) are the field complex amplitudes.

We consider the boundary conditions at the interface for the electric and magnetic fields of the light waves.

We’ll do the perpendicular polarization first.

Fresnel Equations—Parallel electric field

Note that Hecht uses a different notation for the reflected field, which is confusing! Ours is better!

Note that the reflected magnetic field must point into the screen to achieve \( E \times B \propto k \). The \( x \) means “into the screen.”
The math of Mie scattering

- Describes the relationship between incident and scattered electric field components perpendicular and parallel to the scattering plane as observed in the "far-field"

\[
\begin{bmatrix}
E_{\parallel s} \\
E_{\perp s}
\end{bmatrix} = \frac{\exp(-ik(r - z))}{ikr}
\begin{bmatrix}
S_2 & S_3 \\
S_4 & S_1
\end{bmatrix}
\begin{bmatrix}
E_{\parallel i} \\
E_{\perp i}
\end{bmatrix}
\]

- $S$ is known as the "amplitude scattering matrix"
- The total field ($E_{\text{tot}}$) depends on the incident field ($E_i$), the scattered field ($E_s$), and the interaction of these fields ($E_{\text{int}}$). If one observes the scattering from a position which avoids $E_i$, then both $E_s$ and $E_{\text{int}}$ are zero and only $E_i$ is observed.
Amplitude scattering matrix

\[
\begin{bmatrix}
E_{||} \\
E_{\perp}
\end{bmatrix} = \frac{\exp(ik(r - z))}{ikr} \begin{bmatrix}
S_2 & S_3 & E_{||,0} \\
S_4 & S_1 & E_{\perp,0}
\end{bmatrix}
\]

Scattering matrix

- For the “far-field” observation of \(E_s\) at a distance \(r\) and particle of diameter \(d\) such that:

\[
k r \gg n_e^2
\]

\[
k = \frac{2\pi}{\lambda}
\]

\[
n_e = \frac{d}{\lambda}
\]

- Then the scattering elements \(S_3\) and \(S_4\) are equal to zero.

Measured scattering matrix

- Practical experiments measure intensity,

\[
I = \langle EE^* \rangle = (1/2)a^2, \text{where } E = a\exp(-i\delta)
\]

\[
\begin{bmatrix}
I_{lls} \\
I_{\perp s}
\end{bmatrix} = \text{constant} \begin{bmatrix}
|S_2|^2 & 0 \\
0 & |S_1|^2
\end{bmatrix} \begin{bmatrix}
I_{ll} \\
I_{\perp l}
\end{bmatrix}
\]

DEFINING POLARIZATION

\[
p_r = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} = \text{degree of polarization}
\]

\[
p_L = \frac{\sqrt{Q^2 + U^2}}{I} = \text{linear polarized}
\]

\[
p_C = \frac{\sqrt{V^2}}{I} = \text{circular polarized}
\]
Degree of polarization

\[ Pol = \frac{I_{\text{par}} - I_{\text{per}}}{I_{\text{par}} + I_{\text{per}}} \]

\[ I_{\text{par}} = I_0 \exp(-\mu_s L) + fI_{\text{parscatt}} \]

\[ I_{\text{per}} = fI_{\text{perscatt}} \]
Degree of polarisation

\[
\text{Pol} = \frac{\exp(-\mu_L) + f(1 - \exp(-\tau)) / \exp(-\mu_w L) \left\{ 1 + \exp\left( -\frac{\chi \tau}{2} \right) \right\}}{\exp(-\mu_L) + f(1 - \exp(-\tau)) / \exp(-\mu_w L) \left\{ 1 + \exp\left( -\frac{\chi \tau}{2} \right) \right\} + f(1 - \exp(-\tau)) / \exp(-\mu_w L) \left\{ 1 + \exp\left( -\frac{\chi \tau}{2} \right) \right\}}
\]

Pol dependence of \(I_{\text{par}}, I_{\text{per}}\)

Porcine whole blood

Porcine adipose tissue
Birefringence

- Birefringence, or double refraction, is the decomposition of a ray of light into two rays when it passes through certain types of material, depending on the polarization of the light. This effect can occur only if the structure of the material is anisotropic (directionally dependent).
- If the material has a single axis of anisotropy or optical axis, (i.e. it is uniaxial) birefringence can be formalized by assigning two different refractive indices to the material for different polarizations. The birefringence magnitude is then defined by

\[ \Delta n = n_\parallel - n_\perp \]
Imaging skin pathology with polarized light

A. normal

B. freckle

C. pigmented nevus

G. black tattoo

H. nonpigmented intradermal nevus

L. malignant basal cell carcinoma
**System setup**

![Diagram of system setup](image)

**Polarization image**

\[ Pol = \frac{I_{\text{par}} - I_{\text{per}}}{I_{\text{par}} + I_{\text{per}}} \]

\[ I_{\text{par}} = I_0 T_{\text{mel}} \left( R_s + \frac{1}{2} R_d \right) \]

\[ I_{\text{per}} = I_0 T_{\text{mel}} \frac{1}{2} R_d \]

**Prerequisites**

- Glare is about 4% from surface (specular)
- 45% of incident light escapes as random pol
- Melanin act as absorber
- Oriented \( I_{\text{par}} \) and \( I_{\text{per}} \) is 3% and 0% of incident light
- Random \( I_{\text{par}} \) and \( I_{\text{per}} \) is 22.5% and 22.5% of incident light

**Mathematical equations**

\[ Pol = \frac{I_{\text{par}} - I_{\text{per}}}{I_{\text{par}} + I_{\text{per}}} = \frac{I_0 T_{\text{mel}} \left( R_s + \frac{1}{2} R_d \right) - I_0 T_{\text{mel}} \frac{1}{2} R_d}{I_0 T_{\text{mel}} \left( R_s + \frac{1}{2} R_d \right) + I_0 T_{\text{mel}} \frac{1}{2} R_d} = \frac{R_s}{R_s + R_d} \]