Basic Principles of Ultrasound Imaging
System Design

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1 Introduction

Suppose we would like to design an ultrasonic imaging system as shown in Figure 1.

The transducer of width \( a \) sends a short pulse of sonic energy of duration \( \Delta t \) (or \( \Delta r \) assuming a constant velocity of sound \( c \)) into the tissue along a narrow beam (path) of width \( \Delta \theta \). The imaging system then waits a time \( T \) before sending out another pulse. During this time \( T \) any echoes from obstacles in the path are reflected and received by the transducer. Figure 2 illustrates diagrammatically this pulse transmission and reflection.

1.1 Maximum range of an ultrasound imaging system

The velocity of sound in water and tissue (which has essentially the same density as water) is \( c = 1540 \text{m/s} \) (by comparison air at STP is \( c = 331 \text{m/s} \)). If the delay between successive pulses is time \( T \) then the maximum range (axial range) \( r_{\max} \) is

\[
r_{\max} = \frac{cT}{2}
\]

Generally it is desirable to design an ultrasound imaging system (USIS) that has a large axial
range so that deep structures within the body may be imaged. There are two factors that will constrain $r_{\text{max}}$.

- the larger $r_{\text{max}}$ the longer $T$ will need to be.
- energy dissipation. Real tissue is *visco-elastic* and thus longitudinally imparted ultrasound waves will attenuate with distance. Beyond some depth there will not be enough energy returning to the transducer to be detected.

## 2 Axial and angular resolution

The characteristics of the transducer limit the quality of the image which can be produced. Two important characteristics are
• **bandwidth**: The bandwidth will determine the range resolution $\Delta r$. A large bandwidth means a small $\Delta r$ and a small $\Delta r$ means that small differences in the depths of objects can be distinguished.

• **aperture size ($a$)**: A large aperture implies a small $\Delta \theta$. Note that because $\Delta \theta$ is an angular measure (in radians or degrees) the corresponding angular resolution in say millimetres, $\Delta x_\theta$, will vary with depth.

**Problem 1**: What is the explicit relationship between $\Delta \theta$ and $\Delta x_\theta$?

### 2.1 Axial resolution

Typically a transducer (piezo-electric) is capable of generating and detecting frequencies only within some limited range, as illustrated in figure 3.

![Figure 3: Ultrasound transducers operate over a frequency range $B$ centered about a centre (also resonant or carrier) frequency $f_c$.](image)

The nominal centre frequency, $f_c$, is often somewhere between 3 - 10 MHz. What bandwidth is required for a range resolution of $\Delta r$? Let us assume that we require a rectangular pulse (as shown in figure 4) to be transmitted. Note that the pulse is supported by the carrier frequency $f_c$ (i.e the resonant/centre frequency of the transducer).

For a width $\Delta r$ the duration of the pulse will be $\Delta t = \Delta r/c$. By using Fourier analysis the frequencies required to support such a pulse can be obtained and will thus enable the determination of the corresponding bandwidth.

The Fourier transform, $P(f)$, of the pulse in Figure 4 is

$$P(f) = \int_{-\infty}^{\infty} \text{pulse}(t) e^{-i2\pi ft} \, dt$$
\[ = \int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}} \text{pulse}(t) e^{-i2\pi ft} \, dt \]  \quad (2)
\[ = \frac{\Delta t}{2} \text{sinc} [\pi(f_c - f)\Delta t] + \frac{\Delta t}{2} \text{sinc} [\pi(f_c + f)\Delta t] \]  \quad (3)

where \( f \) is frequency and \( \text{sinc} = \frac{\sin(x)}{x} \).

**Problem 2:** Verify equation (3) \( \text{Hint: express cos in complex form.} \)

Figure 4: A pulse of duration \( \Delta t \) supported by a carrier frequency \( f_c \). For convenience this pulse is centred on \( t = 0 \).

The Fourier Transform, \( P(f) \), of the pulse is plotted in Figure 5 and consists of two sinc functions, one centred on \( +f_c \) and the other on \( -f_c \). Most of the energy of the pulse can be transmitted if the transducer has a bandwidth extending to the first zeros of the sinc function.

The first zero of the sinc function occurs when the argument of the sinc function is equal to \( \pi \) and thus

\[ \pi(f_c - f)\Delta t = \pi \]
\[ \Rightarrow (f_c - f)\Delta t = 1 \]
\[ \Rightarrow \Delta f \Delta t = 1 \]
and thus the bandwidth, $B$, is given by

$$B = 2\Delta f = \frac{2}{\Delta t} = \frac{2c}{\Delta r} \quad (4)$$

**Example 1:**

For a radial (axial) resolution, $\Delta r$, of 1 mm what bandwidth is required?

$$\Delta r = 1 \times 10^{-3} \text{ m}$$
$$\Delta t = \frac{\Delta r}{c} \approx 6.5 \times 10^{-7} \text{ s}$$
$$B \approx \frac{2}{\Delta t} = 3.08 \text{ MHz}$$

**Question:**

If this bandwidth is used a rectangular pulse shape will not be obtained. Why? What will the pulse shape look like? Will 1 mm resolution be obtained? Why?

![Figure 5: The Fourier Transform, $P(f)$, of the pulse in Figure 4. Most of the energy of the pulse will be contained within the two main lobes centred at $+f_c$ and $-f_c$.](image)

It is desirable (in order to maximise axial resolution) to increase the bandwidth $B$ by reducing the duration, $\Delta t$, of the pulse. However it is not possible to do so without considering the carrier frequency $f_c$. The pulse width cannot be reduced below $1/f_c$ as there would then be less than one complete cycle of the sinusoid supporting the pulse. Thus
\[ B \leq 2f_c \] (5)

For greater axial resolution a transducer with a higher centre frequency \( f_c \) must be used. In general transducers are resonant systems and their bandwidth is roughly proportional to their centre frequency. In general the bandwidth is typically somewhere between \( 0.2 - 0.5 f_c \).

### 2.2 Angular resolution

Transducers typically present either a circular or rectangular face to the surface of the body. The face of the transducer is the aperture through which the acoustic radiation passes. The aperture may be a single piece of ultrasonic material (e.g. piezo-electric crystal) or be divided up into several pieces. A circular aperture may be divided up into a series of concentric rings, as shown in Figure 6a. This can be used to improve the angular resolution near the transducer by focusing the beam in much the same way as a Fresnel lens. A rectangular array may be divided up into a series of rectangular strips (figure 6b). This approach can be used to electronically steer the beam (see later section on phased arrays).

![Circular and rectangular apertures](image)

(a) circular aperture

(b) rectangular aperture

Figure 6: Typical multi-element transducer array configurations

The size of the aperture determines the size of the beam. Consider an aperture of size \( a \) as shown in figure 7. Further suppose that the entire aperture is driven at a carrier frequency \( f_c \) and transmitting acoustic energy to a receiver at distance \( d \) and angle \( \theta \) with respect to the centre of the aperture. By sweeping the receiver through different angles of \( \theta \) some indication of the width of the beam is obtained. The results is a beam pattern typically like that shown in Figure 8.

![Beam pattern](image)

Ideally for high angular resolution it would be desirable to measure a pattern where most of the acoustic energy is concentrated at \( \theta = 0 \) and very little is concentrated elsewhere. For simple situations like that of figure 6 the beam pattern can be explicitly calculated. The idea
is simple. Each point of the aperture is treated as if it is radiating a circular (or in three dimensions a spherical) wavefront. By the principle of linear superposition the resultant wave at an observation point P is found (see figure 9)

\[
S(t) = \int_{-a/2}^{a/2} s(t - r(y)/c) dy
\]  

where \( s(t) \) is the signal at the aperture. The term \( r(y)/c \) takes into account the time taken for the signal to reach the observation point. Now assume for simplicity that each point in the aperture radiates homogeneously and sinusoidally thus
\[ S(t) = \int_{-a/2}^{a/2} \cos[2\pi f_c (t - r(y)/c)] \, dy \]  

(7)

It is convenient to rewrite the above equation in terms of a complex exponential. Thus writing \( \cos \) as a complex exponential equation (7) becomes

\[
S(t) = \text{Re} \int_{-a/2}^{a/2} e^{-i2\pi f_c (t - r(y)/c)} \, dy = \text{Re} e^{-i2\pi f_c t} \int_{-a/2}^{a/2} e^{-i2\pi f_c r(y)/c} \, dy
\]

where \( \text{Re} \) indicates the real part of the expression. It is useful to separate \( r(y) \) into two parts: a fixed part, \( d \), and a small correction \( \Delta r(y) \)

\[ r(y) = d + \Delta r(y) \]  

(8)

and thus

\[
S(t) = \text{Re} e^{-i2\pi f_c t} \int_{-a/2}^{a/2} e^{-i2\pi f_c (d + \Delta r(y))/c} \, dy
\]

(9)

\[
S(t) = \text{Re} e^{-i2\pi f_c t} e^{-i2\pi f_c d/c} \int_{-a/2}^{a/2} e^{-i2\pi f_c \Delta r(y)/c} \, dy
\]

(10)
Note that the shape of the beam pattern is now embodied in the last term involving the integral (variable geometric term). The other two terms are temporal and constant geometric respectively. These terms can be ignored for the purpose of calculating the beam pattern.

It is possible to work out an explicit form for the function $\Delta r(y)$ for the geometry in Figure 9. For simplicity we will assume that $d \gg a$ so that the angle $\alpha$ is effectively $\pi/2$ (see Figure 10). This is called a “far field” approximation. Thus the variable geometric term, $S'$, reduces to

\[ S' = \int_{-a/2}^{a/2} e^{-i2\pi f_c \sin(\theta)/c} \, dy \]

\[ = a \text{sinc} \left[ \pi f_c a \sin(\theta)/c \right] \]  

(11)

Thus the “far field” ultrasound beam produced by a rectangular aperture has the form of a sinc function as shown in Figure 11. Note that there is one main lobe to the beam and multiple side lobes of decreasing amplitude. This is not the ideal situation. In practice it is desirable to eliminate the side-lobes so that any echoes received can be assumed to arise from a target directly in front of the aperture. In principle and in practice side-lobes can be largely eliminated by using an aperture for which the intensity of the transmitted wave decreases gradually from the centre of the aperture to the edge (figure 12). Using an aperture with a tapered excitation is called *apodisation*. The exact shape of the beam formed depends on the shape used for apodisation. Generally analytic results cannot be obtained and thus numerical and computational (e.g finite element) methods must be used to determine accurately the ultrasound beam pattern.

![Figure 10: “Far field” approximation for calculating ultrasound beam shape.](image-url)
While the rectangular aperture produces a less than ideal beam shape most of the important relationships between beam width and other imaging parameters can be determined. Assume that the effective beam width is is the angular distance between the first zeros of the sinc function on either side of the main lobe. As we have noted previously the first zero is obtained when the argument of the sinc function is equal to $\pi$. Thus from equation (11) and referring to Figure 11 the first zero occurs when

$$\pi f_c a \sin(\Delta \theta)/c = \pi$$

and thus the beam width ($\Delta \theta$) is

$$S' = a \text{sinc}[\pi f_c a \sin(\theta)/c]$$

Figure 11: The ultrasound beam pattern for a rectangular array.

Figure 12: Illustration of an aperture with tapered excitation that reduces side-lobes.
\[ \Delta \theta = 2 \sin^{-1} \left( \frac{c}{a f_c} \right) \]  

(13)

Thus increasing either the aperture size \( a \) and/or the centre frequency \( f_c \) will reduce the beam width and lead to greater angular resolution.

**Example 2:**

For an angular resolution, \( \Delta \theta \), of 1° (i.e \( \pi/180 \) radians) and an aperture no larger than 1 cm what frequency should the USIS operate at?

From equation (13)

\[ f_c = \frac{c}{a \sin(\Delta \theta/2)} \]

and thus

\[ f_c = \frac{1.54 \times 10^3 \text{ m s}^{-1}}{1 \times 10^{-2} \text{ m} \sin(\pi/360)} = 17.7 \text{ MHz} \]

**Problem 3:**

Verify equation equation (11).

**Question:**

How might one go about determining a more accurate relationship between beam shape and the dimensions of the rectangular aperture? Hint: consider the “far field” assumption that was used in obtaining equation (11).

### 2.3 Frame rate

Inside the body there are structures which can move quite rapidly. In order to obtain a clear image of these structures the time taken to acquire the image should be short in comparison to the time scales over which these various structures move. It is therefore important to consider the rate (speed) at which a USIS can acquire individual images.

Consider a USIS that has a field of view \( \theta \) and an angular resolution \( \Delta \theta \) and a maximum range \( r_{\text{max}} \). The number of beams which must be transmitted to interrogate the entire region is \( \theta/\Delta \theta \). The time taken for each beam to cover the distance \( 2r_{\text{max}} \) is \( T = 2r_{\text{max}}/c \) so the total time required to acquire one frame is
\[ t_{\text{frame}} = \frac{2r_{\text{max}}}{c \Delta \theta} \]  
(14)

and thus the frame rate, FR, is simply the reciprocal of the above i.e

\[ \text{FR} = \frac{c \Delta \theta}{2r_{\text{max}} \theta} \]  
(15)

**Example 3:**

Consider an USIS with angular resolution 1° and a field of view of 90°, and an \( r_{\text{max}} \) of 15 cm. What is the frame rate ?

\[ \text{FR} = \frac{1.54 \times 10^3 \text{ m s}^{-1} \times 1}{2 \times 15 \times 10^{-2} \text{ m} \times 90} = 57 \text{ Hz} \]

In practice clinical systems are somewhere around 30 frames per second.

**Question:**

Is 30 frames per second fast enough to image the beating heart ? Who said so ?

### 3 Phased arrays

As we have seen a single transducer will send out an ultrasonic beam only in one direction. In order to form an image the beam must be steered to sweep out a range of directions. This can be easily done mechanically with a motor. The earliest USIS used this approach. It is also possible to use multiple transducers thereby steering the beam electronically by adjusting the relative phase of oscillation of each transducer. This is by far the most common approach currently used and the array of transducers used is call a **phased array**.

The phased array is typically composed of a linear array of \( N \) transducers which together comprise the USIS aperture. For example a USIS aperture of 1 cm filled with 128 transducers means that each transducer is a little less than 0.1 mm in width.

Consider the phased array shown in Figure 13. In this figure there are \( N \) transducers labelled from 0 to \( N - 1 \). The position of centre of the \( n \)-th transducer is

\[ y_n = \frac{a}{n} \left( n + \frac{1 - N}{2} \right) \quad n = 0...N - 1 \]  
(16)

and let the signal fed to the \( n \)-th transducer be
Figure 13: Configuration of transducers in the phased array discussed in the text

\[ s_n(t) = \cos(2\pi f_c t - \phi_n) \]  

(17)

where \( \phi_n \) is the relative phase of each transducer which is adjusted to steer the ultrasonic beam. If \( \phi_n = 0 \) the situation is just the same as that discussed in section 2.2 (Angular resolution) and the centre of the main lobe is centred on \( \theta = 0 \). With the phased array we are able to steer the main lobe of the beam to other values of \( \theta \).

3.1 Phase array analysis for \( N = \infty \)

In practice the number of phase array elements is large, typically of the greater than 100 or so. Thus to first approximation it is reasonable to begin the analysis of the phase array by assuming that the relative phase of adjacent elements is a continuous function of aperture distance \( y \) i.e

\[ \phi_n \sim \phi(y) \]

For this case the signal received at point P is as before given by

\[ S(t) = \int_{-a/2}^{a/2} s(t - r(y)/c) \, dy \]  

(18)

where \( s(t) \) is now defined to be
\[ s(t) = \cos[2\pi f_c t - \phi(y)] \]  

By following the same sequence of steps as in the analysis of section 2.2 we obtain

\[ S(t) = \text{Re} \ e^{-i2\pi f_c t} e^{-i2\pi f_c d/c} \int_{-a/2}^{a/2} e^{-i2\pi f_c \Delta r(y)/c} e^{-i\phi(y)} \, dy \]  

Note that the terms on the right hand side correspond respectively to: temporal, fixed geometric, variable geometric and beam steering components of the ultrasonic beam. Making the “far field” assumption (i.e \( d \gg a \)) the beam pattern (i.e the integral on the right hand side of the above equation) becomes

\[ S' = \int_{-a/2}^{a/2} e^{-i2\pi f_c \{y \sin \theta/c - \phi(y)/(2\pi f_c)\}} \, dy \]  

Note that in equation (11) of section 2.2 \( \phi(y) = 0 \) and the beam had a maximum at \( \theta = 0 \). In this case the ultrasound aperture would be radiating a plane wave as shown in Figure 14(a).

However from Figure 14(b) we observe that a plane wave propagating at an angle \( \theta_0 \) with respect to the transducer normal implies that the difference in the arrival time of a wavefront arriving at \( y \) on a line perpendicular to the transducer normal compared to the arrival time of a wavefront arriving at \( y = 0 \) is given by

\[ \Delta t = \Delta r/c = y \sin(\theta_0)/c \]  

and thus

\[ \phi(y) = \Delta \phi = 2\pi f_c \Delta t = 2\pi f_c y \sin(\theta_0)/c \]  

substituting this expression for \( \phi(y) \) into equation (21) and integrating gives

\[ S' = a \text{sinc}[\pi f_c a (\sin \theta - \sin \theta_0)/c] \]  

As before a sinc function is obtained which has a maximum at \( \theta = \theta_0 \). However as illustrated in Figure 15 it is no longer symmetric about \( \theta = \theta_0 \). Therefore more care is required in calculating the beam width (i.e the angular distance between the first zeros either side of the main lobe). The first zeros lie at

\[ f_c a (\sin \theta - \sin \theta_0)/c = \begin{cases} +1 \\ -1 \end{cases} \]
Figure 14: Diagrammatic illustration of how the central lobe of an ultrasonic beam is steered by linearly varying the phase of transducer oscillation across the USIS aperture. Vertical lines represent wavefronts.

\[
\Delta t = \frac{\Delta r}{c} = a \sin(\theta_0)/c
\]
and thus

\[
\sin \theta_+ = \sin \theta_0 + \frac{c}{a f_c}
\]

\[
\sin \theta_- = \sin \theta_0 - \frac{c}{a f_c}
\]

and therefore the beam width $\Delta \theta$ is

\[
\Delta \theta = \theta_+ - \theta_-
\]

\[
= \sin^{-1} \left( \sin \theta_0 + \frac{c}{a f_c} \right) - \sin^{-1} \left( \sin \theta_0 - \frac{c}{a f_c} \right)
\]

Note that this reduces to equation 13 when $\theta_0 = 0$. Notice also that the beamwidth increases as the beam is steered to the side. This is illustrated in Figure 16.

![Figure 15: Beam pattern of a phase steered array (see equation 24).](image)

Problem 3:

Verify equation (24).
3.2 Phase array analysis for finite $N$

In the previous section the beam pattern obtained assumed an infinite number of infinitely small adjacent transducers. However because real USIS are composed of only a finite number of transducers it is important that we understand the beam patterns produced for such discrete arrays. If the number of transducer elements, $N$, becomes too small, or, if the elements become too far apart there can be problems. Most importantly the beam pattern produced may have more than one main lobe. If this happens, more than one beam is produced and it becomes impossible to determine the radial direction in which an echo was generated.

We can find out exactly when the problem occurs by assuming that the continuous aperture of the previous section now radiates from point sources only i.e

$$s(t) = \frac{a}{N} \sum_{n=0}^{N-1} \cos[2\pi f_c t - \phi_n] \delta(y - y_n)$$  \hspace{1cm} (28)

where $y_n$ is given by equation (16) – see also figure 13. By substituting this into equation (18) and carrying out an identical sequence of steps as performed in the previous section (i.e making the “far field” assumption) the beam pattern for this quantised aperture, $S'_N$, is

$$S'_N = \frac{a}{N} \sum_{n=0}^{N-1} e^{i2\pi f_c y_n(sin \theta - sin \theta_0)/c}$$  \hspace{1cm} (29)
\[ S'_N = \frac{a}{N} \sum_{n=0}^{N-1} e^{i2\pi f_c a (n+1-N)/2} \frac{\sin(\theta - \sin \theta_0) / (Nc)}{(Nc)} \]  

(30)

\[ S'_N = \frac{a}{N} e^{i2\pi f_c a (1-N)(\sin \theta - \sin \theta_0) / (2Nc)} \sum_{n=0}^{N-1} e^{i2\pi f_c n a (\sin \theta - \sin \theta_0) / (Nc)} \]  

(31)

\[ S'_N = \frac{a}{N} e^{i2\pi f_c a (1-N)(\sin \theta - \sin \theta_0) / (2Nc)} \sum_{n=0}^{N-1} e^{ikn} \]  

(32)

where \( k = 2\pi f_c a (\sin \theta - \sin \theta_0) / (Nc) \). Note that the last expression is a geometric series in \( e^{ik} \). Remembering the formula for the sum of a geometric series

\[ \sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x} \]  

(33)

and thus

\[ \sum_{n=0}^{N-1} e^{ikn} = \frac{1 - e^{ikN}}{1 - e^{ik}} \]  

(34)

\[ = \frac{e^{-ikN/2} e^{-ik/2}}{e^{-ikN/2} e^{-ik/2}} \frac{1 - e^{ikN}}{1 - e^{ik}} \]  

(35)

\[ = e^{ik(N-1)/2} \frac{e^{-ikN/2} - e^{ikN/2}}{e^{-ik/2} - e^{ik/2}} \]  

(36)

\[ = e^{ik(N-1)/2} \frac{\sin(kN/2)}{\sin(k/2)} \]  

(37)

Substituting this last result into equation (32) and simplifying gives

\[ S'_N = \frac{a}{N} \frac{\sin[\pi f_c a (\sin \theta - \sin \theta_0) / c]}{\sin[\pi f_c a (\sin \theta - \sin \theta_0) / (Nc)]} \]  

(38)

Note that in the limit as \( N \to \infty \) equation (38) reduces to equation (24). Further note that when

\[ \pi f_c a \frac{\sin \theta - \sin \theta_0}{Nc} = m\pi \quad m = 0, \pm 1, \pm 2, \ldots \]  

(39)

\( S'_N = (-1)^m a/N \) (equation 38). Thus other additional main lobes can appear on either side of the central main lobe as shown in Figure 17 This will cause problems only if these additional lobes lie in the range \(-\pi/2 < \theta < \pi/2\). In particular it can be seen from equation (39) that if

\[ \frac{f_c a}{Nc} < \frac{1}{\max|\sin \theta - \sin \theta_0|} \]  

(40)
then no additional lobes will be seen. As both $\theta$ and $\theta_0$ are constrained to lie in the range $[-\pi/2, \pi/2]$ the maximum of $|\sin \theta - \sin \theta_0| = 2$ and thus no additional lobes will be seen as long as

$$\frac{f_c a}{N_c} < 2 \quad \Rightarrow \frac{a}{N} < \frac{\lambda}{2} \quad (41)$$

where $\lambda = c/f_c$. In other words: additional lobes can be avoided providing the spacing between transducer elements is less than half the ultrasound wavelength.

$$a = 0.01 \text{ m} \; \; f_c = 3 \times 10^6 \text{ Hz} \; \; \theta_0 = 20^\circ \; \; c = 1540 \text{ m s}^{-1}$$

Figure 17: Illustration of the appearance of an additional main lobe for $N = 20 \; (a/N < \lambda/c)$. See text for further details.

Now given finite $N$ how far can we steer the beam, $\theta_0$, before the appearance of additional main lobes? From Figure 17 and equation (39) for $\theta_0 > 0$ we note that an additional lobe can be avoided if

$$\sin^{-1} \left[ \sin \theta_0 - \frac{N_c}{f_c a} \right] \leq -\frac{\pi}{2} \quad (43)$$
or on rearranging

\[ \theta_0 \leq \sin^{-1} \left[ \frac{N_c}{f_c a} - 1 \right] \]  

(44)

and thus for finite \( N \) the field of view will be \( 2 \sin^{-1} \left[ \frac{N_c}{(f_c a) - 1} \right] \) in order to avoid any additional main lobes.

**Example 4:**

Consider the USIS of Example 3: beamwidth = 1°, \( a = 0.01 \) m giving an \( f_c = 17.7 \) MHz. If we want to steer the beam over the range \(-90^\circ - 90^\circ\) what is the minimum number of elements that must be used?

\[ N > \frac{2a}{\lambda} = \frac{2af_c}{c} \]

\[ = \frac{2 \times 1 \times 10^{-2} \text{ m} \times 17.7 \times 10^6 \text{ s}^{-1}}{1540 \text{ m s}^{-1}} \]

\[ \geq 230 \text{ elements} \]

**Example 5:**

Consider instead we want to use 128 elements and not 230 as in the previous example. How far can we steer the beam without generating additional, and therefore unwanted, main lobes? What is the field of view for this USIS.

\[ \theta_0 \leq \sin^{-1} \left[ \frac{128 \times 1540 \text{ m s}^{-1}}{17.7 \times 10^6 \text{ s}^{-1} \times 1 \times 10^{-2} \text{ m}} - 1 \right] \]

\[ \leq 0.11 \text{ rad (6.53°)} \]

and therefore the field of view is 0.22 radians or about 13.06°, which is not very good. Note that the field of view can be increased by reducing the linear dimension of the USIS aperture \( a \).

**Problem 4:**

Elaborate the steps required to obtain equation (38).
4 Summary

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum range, $r_{\text{max}}$</td>
<td>$r_{\text{max}} = \frac{cT}{2}$</td>
</tr>
<tr>
<td>bandwidth, $B$</td>
<td>$B = \frac{2}{\Delta t} = \frac{2c}{\Delta r} &lt; 2f_c$</td>
</tr>
<tr>
<td>frame rate, FR</td>
<td>$\text{FR} = \frac{c\Delta \theta}{2r_{\text{max}} \theta}$</td>
</tr>
<tr>
<td>beamwidth, $\Delta \theta$</td>
<td>$\Delta \theta = \sin^{-1}\left(\sin \theta_0 + \frac{c}{af_c}\right) - \sin^{-1}\left(\sin \theta_0 - \frac{c}{af_c}\right)$</td>
</tr>
<tr>
<td>number of elements</td>
<td>$N &gt; \frac{2a}{\lambda}$</td>
</tr>
<tr>
<td>beamsteer</td>
<td>$\theta_0 \leq \sin^{-1}\left[\frac{Nc}{f_c a} - 1\right]$</td>
</tr>
</tbody>
</table>

Table 1: Summary of the main equation required to design a USIS with a rectangular aperture