Solutions to exam in
Medical Image Analysis - TBMI45
2008-01-07

The solutions to some of the assignments can here be presented more detailed than needed on the exam.

Part 1

1. By registration of the image to a segmented target image, segmentation and registration is performed at the same time. This can be very useful, for example if you want to segment functional areas of the brain.

2. According to the sample theorem a sample density of \( \Delta k \leq \frac{1}{\pi} = 0.01 \) needed.

   This sample density gives us the distance \( \sqrt{2} \Delta k \) to origo.

3. a). False, they are region based, se lecture notes on Snakes and Watershed.
   b). False, the internal forces must always point to the middle of the object, otherways the contour will get stuck in convex coverings (for example that the contour does not go in between the fingers of a hand).
   c). True, some work afterwards is often needed.
   d). False, it comes from the level set method.

4. I). If the external energy is defined as \( E_{ext} = I \) it gives the external forces \( F_{ext} = \nabla I \), i.e. the gradient of the image.

   An edge will then give a force field that points in the same direction around the edge and these forces will point in different directions depending on if the transition is from dark to light or light to dark.

II). If the external energy is defined as \( E_{ext} = |\nabla I| = \sqrt{(\nabla_x I)^2 + (\nabla_y I)^2} \) it will give external forces that always point in towards the edge. See the image in the lesson material.

   The energy in I). is sensitive for the sign at the edge. This means that we should know if the object to be segmented is light compared to the background or vice versa. For this choice of external energy there is also a cost for weak linear transitions in the image, the contour can converge to something that is not a distinct edge.

   The energy in II). is a second derivative and will not be sensitive for weak linear transitions and will thus converge faster to distinct edges. It is insensitive to the sign of the transition and creates a potential hole at the edges, which means that the contour can find the solution from arbitrary side of the edge if the internal forces allow the contour to move in both directions.
5. The weighted (as well as the unweighted) distortion will be zero and is independent of \( \alpha \) since the number of samples is equal in the both domains.

The Fourier domain for a sampled signal is continuous and even if the error is zero in the sample points there can be large errors between the sample points if the distance between the sample points is large. It is therefore important to have an adequate resolution when you sample in the Fourier domain. In the practical case you should use at least twice as many samples in the Fourier domain as in the spatial domain.

6. Linus solves the problem by flipping the filter and sets the new filter to \( f'(x) = f(-x) \).

Linnea has thought more of the problem and realizes that it is the sign of the imaginary part of \( f(x) \) that should be changed, thus she takes the complex conjugate of the filter \( f'(x) = f(x)^* \).

The reason that both ways work is that the real part of a quadrature filter is even and the imaginary part is odd.

7. The correct answer is m-func 3. An overshoot on the m-func \((\alpha > 0)\) gives a compression of the dynamics of the signal. (The linearly increasing amplitude would get a convex form.) Thereby we can exclude m-func 1 and 2. The easiest way to make the choice between m-func 3 and 4 is by looking at the noise threshold \( \sigma \). If m-func 4 was used the area where the original signal has a magnitude below 0.1 would have been multiplied with a constant that is close to zero, which does not agree with the result of the filtering.

8. Expand the function \( f(x) \):
\[
f(x) = 4 - (x - 2)^2 = 4 - (x^2 - 4x + 4) = 4x - x^2
\]
This function can be expressed exactly in the base of \( x \) and \( x^2 \) with the coefficients 4 and -1. The error will be zero.

9. If the object has the dampening function \( f(x, \Theta) \) the signal strength will in the first case be
\[
R(x, \Theta) = R_0 e^{- \int f(x, \Theta) \, dx} = e^{- \int f(x, \Theta) \, dx}
\]
With the new energy the signal strength will be
\[
R_2(x, \Theta) = e^{-1.2 \int f(x, \Theta) \, dx} = R(x, \Theta)^{1.2}
\]

10. The distance between samples along a line in the Fourier domain becomes \( \Delta u = 1/0.7 \, [m^{-1}] \)
Sample density density along a line: \( 1/\Delta u = 0.7 \, [m] \).
Sample density in the angle direction: \( N/2\pi \left[ \frac{\text{linjer}}{\text{rad}} \right] \)
Radial dependency since the number of lines is constant: \(|u|^{-1} \, [m] \)
11. a) Even if you disregard the noise, the signal that you measure does not exactly look like the paradigm, one difference is that it is delayed (due to the BOLD delay). To get high correlation despite such differences you can for example correlate the intensity variation in a pixel with a set of sine- and cosine-functions instead of with the paradigm. A linear combination of a sine- and cosine-function can produce an arbitrary phase of the signal and thereby find the unknown delay.

b) If you use too many different basis functions, these can be combined such that the result will be similar to anything (for example noise). This will lead to high correlation even where there is no brain activity.

12. The sample frequency $f_s = 20$MHz means that the maximal frequency $\pi$ that can be represented in the sampled system is 10MHz. The plots show the spectra in the area $[-\pi/2, \pi/2]$. The maximal frequency in the plots is thereby 5MHz.

a) Spectra 3. The signal $f(t) = \cos(2\pi f_0 t)$ will correspond to two spikes with $1.66/5 = 1/3$ of the maximal frequency. The reason that the spectra is not shaped as two spikes depends on the cos-window.

b) Spectra 4. The received echo is amplitude modulated which gives a widening of the spectra.

c) Spectra 2. The quadrature filter eliminates the negative frequency components. The center frequency $\omega_c$ coincides with $f_0$ and a bandwidth of 2 octaves corresponds well to the shape of the spectra.

d) Spectra 1. By computing the absolute value $|q(t)|$ we will get a phase invariant (amplitude demodulated) signal that consists of low frequencies.
Part 2

13. a). A global translation gives the transformation model \( v(x) = B(x)p = \Delta x \).
   The equation for optical flow will then give an over determined equation system
   \[
   \nabla I_2(x) \Delta x \approx I_1(x) - I_2(x)
   \]
   with the solution
   \[
   \Delta x = \left( \nabla I_2^T \nabla I_2 \right)^{-1} \begin{bmatrix}
   \nabla I_2^T (I_1(x) - I_2(x)) \\
   A b
   \end{bmatrix}
   \]
   \[
   A = \left( \begin{array}{cc}
   \sum (\nabla_x I_2)^2 & \sum \nabla_x I_2 \nabla_y I_2 \\
   \sum \nabla_x I_2 \nabla_y I_2 & \sum (\nabla_y I_2)^2
   \end{array} \right)^{-1}
   \]
   \[
   b = \left( \begin{array}{c}
   \sum (I_1 - I_2) \nabla_x I_2 \\
   \sum (I_1 - I_2) \nabla_y I_2
   \end{array} \right)
   \]
   which gives
   \[
   \Delta x = \begin{bmatrix}
   1.0 \\
   -0.72
   \end{bmatrix}
   \]
   b). Bilinear interpolation gives \((I_2(x + \Delta x) - I_1(x))^2 \approx 7.5e - 3\).

14. a) The trajectory is given by the integral of Gx and Gy, see figure.
You can see from a) that k-space will be searched with the same distance over the whole area, thus we have the 'normal situation' and the least time that is needed is then
\[ 256 \times 256 \times 4 = 262 \text{ [mS]} \]

15. The local tensor for the environment is calculated by averaging:

\[ T = \frac{1}{2} (T_1 + T_2) = \frac{1}{25} e_1 e_1^T + \alpha e_2 e_2^T \quad e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

The control tensor has the same eigensystem as the local tensor

\[ C = \gamma_1 e_1 e_1^T + \gamma_1 \mu_2 e_2 e_2^T \]

a) The value of \( \mu_2 \) is given by the \( \mu \)-function. A maximally anisotrop tensor, \( \mu_2 = 0 \), means that \( \lambda_2 / \lambda_1 \leq 0.5 \) for the local tensor. We have two possibilities: either \( \alpha = \lambda_2 \leq 0.5 \lambda_1 = \frac{1}{50} \) or \( \alpha = \lambda_1 \geq 2 \lambda_2 = \frac{4}{50} \).

Answer: \( \alpha \leq \frac{1}{50} \) and \( \alpha \geq \frac{4}{50} \) give a maximally anisotrop control tensor.

b) Since \( \mu_2 = 0 \) the magnitude of the control tensor is only determined by \( \gamma_1 \), \( m\text{-func}(0.15) \) gives \( \gamma_1 = 0.8 \)

\[ \sqrt{\alpha^2 + \left( \frac{1}{25} \right)^2} = 0.15 \text{ gives } \alpha = \frac{0.145}{\sqrt{0.15}} \]

If \( \alpha \) is set to 0.145 the control tensor will be maximally anisotrop and gets \( \| C \| = 0.8 \)

16. a) The filter should be a quadrature filter with center frequency \( \omega_c \approx 0.5 \) and bandwidth such most of the signal will pass.

The convolution will make sure that \( q(t) \) does not have any negative frequencies, since the filter is zero for these. Otherwise the spectra has not changed much except that the signal has been dampened by a narrow bandpass filter.

The absolute value \( |q(t)| \) corresponds to a convolution with itself in the Fourier domain, which in the temporal domain is equal to get the envelop since the carrier wave disappears.

b) The mapping is not monotonous, meaning that some areas in the image will be inverted when light pixels and dark pixels are mixed. For example \( T(0.5) \approx 0.55 \) but \( T(1) = 0.27 \) meaning that gray pixels get a little lighter, but completely white pixels get dark gray.