Answers to the exam in
TBMI02 - medical image analysis 2015-01-17

The answers presented below may be more exhaustive compared to what is normally required at the exam.

Part 1

1. k-space sampling number \(2\) was used for the middle image. k-space sampling number \(4\) was used for the right image.

The middle image is aliased in the spatial domain. Note that apart from the aliasing effects the size of the head is bigger compared to the correct image. The sampling distance in k-space is about \(1.5\) times too big for the object (compare e.g. the distance between the eyes in the left and the middle image).

For the right image the field of view is right. This implies that the sampling distance in k-space is correct. The image is too smooth which implies that the actual sampling has not reached sufficiently far out in k-space.

2. \[
\min \epsilon^2 = \sum_{x=-1}^{x=1} [w(x) (b_1 2^x + b_2 2^{-x} - 1 - 0.25x^2)]^2 \quad w(x) = [1, 0.5, 1]
\]

Since both the objective function and the weight function are even we have \(b_1(x) = b_2(-x)\) implying that that \(b_1 = b_2\).

\[
\epsilon^2 = (2.5 b_1 - 1.25)^2 + 0.25 (2 b_1 - 1)^2 + (2.5 b_1 - 1.25)^2
\]

It’s simple to see that the answer is: \(b_1 = b_2 = 0.5\) (Giving \(\epsilon^2 = 0\))

Or if you prefer, set \(\frac{d\epsilon^2}{db_1} = 0\) and solve.

3. a) The problem is that the distance between the object in the two images is too large for the used quadrature filters. The filters must for example be able to “see” the same edge in both images, to calculate the phase difference which determines how to move one of the images. An appropriate spatial size for a quadrature filter with a center frequency of \(\pi/4\) is something like \(13 \times 13\) or \(15 \times 15\) pixels. Such a filter can clearly not work for cases where the distance is 25 pixels.

b) There are basically two solutions to the problem; to use filters with a lower center frequency or to downsample the images. Filters with a lower center frequency need to be larger in the spatial domain, and it will therefore take more time to apply the filters (the solution is not efficient). Downsampling the images a factor 2 in each dimension will reduce the distance between the object in image 1 and image 2 to 12.5 pixels. This distance is still too large for the used quadrature filters. Downsampling the images a factor 4 in each dimension will reduce the distance to 6.25 pixels, which may still be too large. Downsampling the images a factor 8 in each dimension will give a distance of 3.125 pixels, which is small enough for the used filters. Once a solution has been found for this scale, the registration can continue on finer scales, to improve the solution.
4. Watershed is a border based segmentation method.

False, it is region based.

It is impossible to perform segmentation by using a registration algorithm.

False, it can for example be done by registering a volume to an atlas with presegmented areas.

To avoid clustering in the active contours algorithm, decreasing the number of nodes is a good idea.

True, the risk of clustering is lower with a smaller number of nodes.

It is hard to change the topology of an active contour.

True, but it is easy for a level set.

5. a) The Fourier domain is periodic with a period of $2\pi$. For $N=3$

$$u = [-2\pi/3, 0, 2\pi/3] (+n 2\pi)$$

gives a uniform sampling that is symmetric and includes the origin.

b) 

$$u = \sum_{k=-N/2}^{N/2} k \frac{2\pi}{N}$$

6. The $L_2$ norm is attractive because the error function is then strictly convex. A nice property of convex functions is that any local minimum is also a global minimum. A minimum can for the $L_2$ norm easily be found by taking the derivative of the error function, with respect to the parameter vector, setting the derivative to 0 and then solving the resulting equation system. The error function for a $L_1$ norm is also convex, but it is not as easy to find a minimum.

7. The brain in the fMRI data can easily be segmented by for example using a thresholding approach. The voxels outside the brain are black and thereby have a very low intensity, while voxels in the brain have a much higher intensity. Göran can then perform the statistical calculations for the brain voxels only.

8. You are looking at the plots of an optimized quadrature filter. What properties should you look for in the Fourier domain and in the Spatial domain when you judge the result of the optimization from these plots?

**Fourier domain:** The function should be smooth. Be zero value in the origin. Have a peak value of 1 for the expected center frequency. The plot should only have nonzero values in one half of the FD to provide a phase independent magnitude estimate.

**Spatial domain:** The function should be smooth and approach zero at the border for both the real and the imaginary part. No large coefficients close the border.
9. Decide if the following statements are true or false. Three right answers gives 1p, four right answers gives 2p.

<table>
<thead>
<tr>
<th></th>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Amplification of noise induced structures are mainly due to errors in the ( m )-func mapping.</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>2. The ( \mu )-function is high in areas that just contain noise.</td>
<td>x</td>
<td></td>
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<tr>
<td>3. The LP-filter in the adaptive filtering is not used when ( | C | &gt; 0 ).</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>4. In 2D the local tensor and the GOP-representation contain the same information.</td>
<td></td>
<td>x</td>
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</tbody>
</table>

10. The brain activity is detected from the intensity variations over time for the pixels in an MR-image. If the subject moves the head during the acquisition the intensity of the pixels will correspond to a different area of the brain due to the motion. By applying a registration to the fMRI-images the pixel intensities will correspond to the same area of the brain for the entire experiment. Only when this condition is fulfilled we may decide if this part of the brain is active or not.

11.

<table>
<thead>
<tr>
<th></th>
<th>True</th>
<th>False</th>
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<tbody>
<tr>
<td>1. The ( \gamma_1 )-image has low magnitude in areas that just contain noise.</td>
<td>x</td>
<td></td>
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<tr>
<td>2. The ( \mu_2 ) mapping is independent of ( | T | ).</td>
<td>x</td>
<td></td>
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<tr>
<td>3. If ( | C | = 0 ) only the lowpass filter is used.</td>
<td>x</td>
<td></td>
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<tr>
<td>4. If the SNR in the image is increased the noise threshold in the ( m )-function should be decreased.</td>
<td>x</td>
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12.

- The spin vectors will be oriented in parallel or anti-parallel to the stationary magnetic field and start to precess around the axis of the magnetic field with the Larmor frequency \( \omega = \gamma B_0 \) where \( \gamma = 42.58 \text{ MHz/T} \).
- For an RF-pulse with the same frequency as the Larmor frequency \( 127.7 = 42.58 B_0 \) MHz the spin vectors will move away from the z-axis and rotate in a spiral towards the xy-plane.

b) With a gradient \( G_x \) only the spin vectors for the den yz-slice within the object that corresponds to \( x = 0 \) will be activated by an electromagnetic pulse with a frequency of 127.7MHz.
13. a) Different tissues have different relaxation rates ($T_1$ and $T_2$). A good contrast means that different tissues will have different intensities in the MR image. If we sample directly after the RF pulse, the signal will be very strong but there will not be any contrast between different tissues, since there has not been any time for the different relaxation rates to matter.

b) 
\[ k_x(t) = \frac{\gamma}{2\pi} \int_0^1 G_x(\tau) d\tau \]
\[ k_y(t) = \frac{\gamma}{2\pi} \int_0^1 G_y(\tau) d\tau \]

c) The scan pattern becomes “MRI”
d) EPI sampling works by using a single excitation for the whole image, compared to using one excitation per line. EPI sampling is much faster compared to conventional sampling, but the image quality is much lower.

14. a) The first assumption is that the image intensity does not change between the two images (i.e. that the only difference is a movement). This can be written as

\[ I(x, t) = I(x + \Delta x, t + 1) \quad (1) \]

The second assumption is that the image locally can be described as a slanted plane, i.e. with a first order Taylor expansion. This can be written as

\[ I(x + v(x), t + 1) = I(x, t + 1) + \nabla I^T v \quad (2) \]

where \( \nabla I = [\nabla_x I, \nabla_y I]^T \). Use the first assumption

\[ I(x + \Delta x, y + \Delta y, t + \Delta t) = I(x, y, t) \quad (3) \]

Using the other assumption, we apply a first order Taylor expansion to the expression above and get

\[ \Delta x \nabla_x I + \Delta y \nabla_y I - \Delta t (I_1 - I_2) = 0 \quad (4) \]

where \( I_2 \) is the second image and \( I_1 \) is the first image and \( I_1 - I_2 \) is an estimate of the time derivative. If we divide each term by \( \Delta t \) we get

\[ \frac{\Delta x}{\Delta t} \nabla_x I + \frac{\Delta y}{\Delta t} \nabla_y I - (I_1 - I_2) = 0 \quad (5) \]

and since \( \frac{\Delta x}{\Delta t} \) is the velocity in the x-direction \( v_x \) and \( \frac{\Delta y}{\Delta t} \) is the velocity in the y-direction \( v_y \) we finally get the classical flow equation

\[ v_x \nabla_x I + v_y \nabla_y I - (I_1 - I_2) = 0 \quad (6) \]

\( \nabla_x I \) is here the derivative of the image in the x-direction and \( \nabla_y I \) is the derivative of the image in the y-direction. A more compact way to write this is

\[ \nabla I^T v - \Delta I = 0 \quad (7) \]
where $\Delta I = I(x, t) - I(x, t + 1) = I_1 - I_2$.

b) In each iteration of the algorithm interpolation is used to get the image at the new location. Interpolation has a similar effect as lowpass filtering and a repeated lowpass filtering will lead to a result with a very low quality. If $p$ instead is updated in each iteration, the interpolation will always be performed from the original image and the lowpass effect will thereby only occur once.

c) The assumptions needed for optical flow are better suited for the local phase.

15. A wave pattern with a wavelength $\lambda$ corresponds to the frequency

$$u = \frac{2\pi}{\lambda}$$

where the Nyqvist frequency, $\pi$, has a wavelength of 2 pixels.

The relative Bandwidth $\beta$ of the lognormal radial function

$$R(\rho) = e^{-\frac{4}{\pi^2 \ln(2)} \ln^2 \left( \frac{\rho}{u_0} \right)}$$

where $\rho = \|u\|$ is defined for $R(\rho) = 0$ which is equivalent to a drop in magnitude of $-20 \log(0.5) = 6\text{dB}$.

The relative bandwidth is computed as

$$\beta = \log_2 \left( \frac{u_h}{u_l} \right) = 2^\beta = 2$$

The center frequency is the geometric mean of $u_l$ and $u_h$

$$u_0 = \sqrt{u_l u_h}$$

a) $u_0 = \pi/3$ gives $u_l = \frac{\pi}{3\sqrt{2}}$ and $u_h = \frac{\pi \sqrt{2}}{3}$ which corresponds to a wavelength range of $\lambda = [4.2 - 8.5]$ pixels.

b) $u_0 = \pi/6$ gives $u_l = \frac{\pi}{6\sqrt{2}}$ and $u_h = \frac{\pi \sqrt{2}}{6}$ which corresponds to a wavelength range of $\lambda = [8.5 - 17]$ pixels.

c) The mapping between the local structure tensor $T = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T$ to the control tensor $C = \gamma_1 e_1 e_1^T + \gamma_2 e_2 e_2^T$ is performed using the $m$-function and the $\mu$-function.

The $m$-function is a magnitude mapping of the normalized tensor magnitude $\|T\|/\|T\|_{\text{max}}$ to $\gamma_1$.

The $\mu$-function is a shape mapping from $\lambda_2/\lambda_1$ to $\mu_2 = \gamma_2/\gamma_1$.

We start by computing $\lambda_1$ and $\lambda_2$ and $\|T\| = \sqrt{\lambda_1^2 + \lambda_2^2}$ for all three tensors.

$$T_1: \quad \lambda_1 = \frac{2}{\sqrt{5}} \quad \lambda_2 = \frac{1}{\sqrt{5}} \quad \|T_1\| = 1 \quad \frac{\lambda_2}{\lambda_1} = \frac{1}{2}$$

$$T_2: \quad \lambda_1 = \frac{3}{10\sqrt{2}} \quad \lambda_2 = \frac{3}{10\sqrt{2}} \quad \|T_2\| = \frac{3}{10} \quad \frac{\lambda_2}{\lambda_1} = 1$$

$$T_3: \quad \lambda_1 = \frac{3}{\sqrt{17}} \quad \lambda_2 = \frac{3}{4\sqrt{17}} \quad \|T_3\| = \frac{3}{4} \quad \frac{\lambda_2}{\lambda_1} = \frac{1}{4}$$

The next step is to estimate $\gamma_1$ and $\gamma_2$ from the FD representation of the adaptive filters in the figures. These adaptive filters are a weighted sum of all the BP filters and the LP filter. $\gamma_1$ correspond to the magnitude of the BP filter part in the main direction, $e_1$, and $\gamma_2$ correspond to the magnitude of the BP filters in the $e_2$ direction. For the three different filter shapes we get:
\begin{align*}
\mathbf{C}_1 : & \quad \gamma_1 = 1 \quad \gamma_2 = 0.5 \quad \mu_2 = \frac{\gamma_2}{\gamma_1} = 0.5 \\
\mathbf{C}_2 : & \quad \gamma_1 = 0 \quad \gamma_2 = 0 \quad \mu_2 = \frac{\gamma_2}{\gamma_1} = \text{undefined} \\
\mathbf{C}_3 : & \quad \gamma_1 = 1 \quad \gamma_2 = 0 \quad \mu_2 = \frac{\gamma_2}{\gamma_1} = 0
\end{align*}

The \( \mu_2 \)-value of \( \mathbf{C}_2 \) is undefined since only the LP-filter is used in filter shape 2.

The \( m \)-function should pass through \((0.3, 0.0), (0.75, 1) \) and \((1.0, 1.0)\).

The \( \mu \)-function should pass through the points \((0.25, 0), (0.5, 0.5)\).

16. a) For a tensor:

\[
\mathbf{T} = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}
\]

The Frobenius norm is computed as the square root of the sum of the tensor elements to the power of two:

\[
\|\mathbf{T}\| = \sqrt{t_1^2 + 2t_2^2 + t_3^2} = \sqrt{\lambda_1^2 + \lambda_2^2} = 2 \sqrt{13} \approx 7.21
\]

The vector representation, \( \mathbf{z} \), (GOP-representation) is computed as:

\[
\mathbf{z} = t_3 - t_1 + 2t_3 = -2i
\]

b) There many ways to decompose \( \mathbf{T} \) into a sum of rank 1 tensors. One way is to use the relation

\[
\mathbf{T} = \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^T \quad \mathbf{T}_1 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \quad \mathbf{T}_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}
\]

where \( \mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), \( \mathbf{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), \( \lambda_1 = 6 \) and \( \lambda_2 = 4 \)

This way \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \) share the same eigen system but that is not required in the question.

Consequently there exists infinitely many possibilities to decompose \( \mathbf{T} \) into a sum of rank 1 tensors.

For this choice we get

\[
\|\mathbf{T}_1\| = 6 \quad \mathbf{z}_1 = -6i \quad \|\mathbf{T}_2\| = 4 \quad \mathbf{z}_2 = 4i
\]

We conclude that \( \|\mathbf{T}_1\| = |\mathbf{z}_1| \), \( \|\mathbf{T}_2\| = |\mathbf{z}_2| \) and \( \mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2 \). This is true for all rank 1 decompositions of \( \mathbf{T} \).