Answers to the exam in
Medical image analysis - TBMI02
2012-08-24

The answers presented below may be more exhaustive compared to what is normally required at the exam.

Part 1

1. \( \min \epsilon^2 = \sum_{x=-1}^{x=1} [w(x) (\alpha 2^x + \beta 2^{-x} - 1 - 0.25x^2)]^2\quad w(x) = [1, 2, 1] \)

   Since both the objective function and the weight function are even we have \( b_1(x) = b_2(-x) \)
implying that \( \alpha = \beta. \)

   \( \epsilon^2 = (2.5 \alpha - 1.25)^2 + 4(2 \alpha - 1)^2 + (2.5 \alpha - 1.25)^2 \)

   It’s simple to see that the answer is: \( \alpha = \beta = 0.5 \) (Giving \( \epsilon^2 = 0 \))

   Or if you prefer, set \( \frac{d\epsilon^2}{d\alpha} = 0 \) and solve.

2. To ensure that the object does not get aliased in the reconstruction the maximum distance between two adjacent samples in k-space is \( \frac{2\pi}{100} \). The distance to a k-space corner is then:

   \( \frac{2\pi}{100} \sqrt{(0.5 \times 300)^2 + (0.5 \times 400)^2} = 5\pi \)

3. a) The locations where there are significant spatial gradients in the T2 volume.

   b) The artifacts can be removed by performing a registration of the volumes from different times before doing the temporal fMRI analysis.

4. Edge based example - Live wire

   Region based example - Watershed

   There are an infinite number of cases where the results will differ. One example is:

   \( f(x, y) = 10e^{-4\log(x^2+y^2)^2} + \text{sign}(y) \)

   Live wire will pick up the edge at \( y = 0 \) but Watershed will see the circular ring made by the radial lognorm function and result in a circle with radius 1.

5. a) To image large spatial areas, the samples in k-space must be put close together.

   b) To get images with high resolution the samples need to cover k-space up to high frequencies, i.e. samples are needed far away from origo.
6. The easiest way might be to use region growing. Start in the point given by the mouse pointer and study its neighbours. If they are similar enough compared to the first point they are also allowed to be a part of the object. Then study their neighbours and determine if they are similar enough to their neighbours. Continue like this and set the border of the object where the neighbouring pixels are too different. This method is easy and fast and it is easy to understand the meaning of the parameters. It is though rather sensitive to noise (which the Magic Wand in Photoshop also is). This problem could be solved by some noise reduction algorithm, for example a simple lowpass filtering.

7. a) The local phase is a better representation since it is invariant to the intensity in the image and it can thereby handle registration of images with different intensity. The local phase is better suited for the assumptions that are necessary in the optical flow algorithm.
   
   b) The local phase from quadrature filters describe a signal only for a certain direction. Therefore it is necessary to use a certainty that tells us which filters to trust in each point. It is also important to only use the phase if the magnitude of the filter response is high, otherwise the phase is more or less noise.
   
   Since the phase is periodic with $2\pi$, and the fact that the uncertainty increases with the phase difference, it is also important to give a low weight to big phase differences (a big phase difference indicates that the filters have picked up different things in the reference image and in the altered image).

8. It is most critical that the deviation between the optimized filter and the ideal frequency response is small for the most common frequencies in the image.

   You can compute the spectrum of the image and use as a frequency weight. For most natural images the spectrum can be modeled as $u^{-\alpha}$ where $0.5 \leq \alpha \leq 1.5$

9. Problem 1

   The problem is that the internal forces have been defined such that the arc length of the contour will be minimized. This means that the snake will get stuck in convex hulls, i.e. it does not want to grow inwards since this increases the length of the contour. The most simple solution is to make the internal force always point inwards, then the snake will be able to handle these areas as well.

   Problem 2

   The knot on the contour is a result of that the nodes have moved closer and closer to each other and finally crossed. The contour has been turned inside out and the snake grows in the wrong direction. To prevent this from happening some kind of cluster elimination should be included. This can be done by moving the nodes in each iteration such that they always lie on a certain distance from each other, or to resample the curve and remove or add nodes in the areas where it is necessary.

10. The Fourier domain of a sampled signal is a continuous function. In the optimization we only observe this function in the sample points. If the distance between the sample points is large there may occur large errors inbetween the samples that are not detected in the optimization. To optimize a filter using an equal number of samples in both domains is not meaningful as the ideal frequency function then can be realised exactly in those points. In practice you need twice the number of samples (in each dimension) in the Fourier domain.
11. The correct answer is m-func 2. The result of the adaptive filtering shows that the low amplitude part of the signal that exceeds the noise threshold is amplified more than the high amplitude part of the signal. This implies that the m-func has a rather significant overshoot (resulting in a compression of the signal). This observation rules out m-func 3 and 4. The easiest way to make the choice between m-func 1 and 2 is by looking at the noise threshold $\sigma$. If m-func 1 was used the part where the original signal has a magnitude above 0.05 would have been preserved which does not agree with the result of the filtering.

12. 

$$
T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix}
$$

$$
z(\varphi) = t_1 - t_3 + i2t_2 = 2i
$$

for both tensors.

The difference between $T_1$ and $T_2$ is isotropic

$$
T_2 - T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

This part has no orientation information as $\lambda_1 = \lambda_2$ and the eigenvectors can be chosen arbitrarily. A small isotropic contribution can e.g. occur if noise is added to the image.
Part 2

13. a) The parameter measured in fMRI is $T^*_2$. The dynamics is usually assumed to be an exponential decay, i.e. the time dependence of the magnetisation, $M$ (measured signal strength) is given by:

$$M(t) \propto e^{-(t/T^*_2)}$$

b) The frequency at which a proton spins inside a voxel rotates is not identical for all protons. This implies that in time different protons will have different phase angles, a process known as dephasing. The measured signal is the sum of the magnetisation due to all protons in the voxel. Using the spin-vector model this means that vectors of different orientations will be summed and the more the orientations differ the shorter the resultant will be.

14. The local tensor for the neighbourhood is computed by averaging:

$$T = \frac{1}{2} (T_1 + T_2) = \mathbf{e}_1 \mathbf{e}_1^T + \alpha \mathbf{e}_2 \mathbf{e}_2^T$$

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The control tensor and the local tensor have identical eigen vectors.

$$\mathbf{C} = \gamma_1 \mathbf{e}_1 \mathbf{e}_1^T + \gamma_1 \mu_2 \mathbf{e}_2 \mathbf{e}_2^T$$

a) The control tensor is isotropic for $\mu_2 = 1$. From the $\mu$-function we get $\lambda_2/\lambda_1 \geq 0.9$ results in an isotropic control tensor. Choose $\alpha = \lambda_2 = 0.9$

b) $\gamma_1 = m$-func $\left(\sqrt{\lambda^2_1 + \lambda^2_2/14}\right) = m$-func$(0.1) = 0.5$

$$\|\mathbf{C}\| = \sqrt{\gamma^2_1 + (\gamma_1 \mu_2)^2} = 0.71$$

15. a) $I_1$ and $I_2$ are volumes with scalar valued intensities. The intensity gradient is thereby a three dimensional vector in each point, as the motion field $\mathbf{r}(\mathbf{x})$. The motion field is described by parameters with the help of a base matrix $\mathbf{B}(\mathbf{x})$ and a parameter vector $\mathbf{p}$, $\mathbf{r}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{p}$. A translation is coded with 3 parameters since there are 3 dimensions,

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

b) Assume that the rotation takes place around one axis, for example the z-axis. The rotation matrix then becomes

$$\mathbf{R} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
motion field need two parameters for the rotation. Observe that a rotation matrix
rotates coordinates while the motion field tells how things should be moved.

\[
\mathbf{r}(\mathbf{x}) = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} p_4 & p_5 & 0 \\ -p_5 & p_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & y & -x \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix}
\]

16. A line width of 1.5-8.5 pixels corresponds to \( u_l = \pi/8.5 \) and \( u_h = \pi/1.5 \) for the quadrature filter.

a) The Bandwidth \( B = \log_2(8.5/1.5) = 2.5 \) octaves.

b) The centre frequency \( u_c = \sqrt{\pi/1.5 \pi/8.5} \approx \pi/3.6 \) has a wavelength of \( \lambda = 7 \) pixels.
   It is reasonable to assume that a this filter require a spatial support of that is 1.5-2 times larger than the wavelength of the centre frequency. Try an odd size in the range of 11-15 pixels.

c) Compute \( \lambda_1 \) and \( \lambda_2 \) for all three tensors.

\[
\begin{align*}
T1 : & \quad \lambda_1 = \frac{1}{\sqrt{5}} \quad \lambda_2 = \frac{1}{2\sqrt{5}} \quad \sqrt{\lambda_1^2 + \lambda_2^2} = \frac{1}{2} \quad \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \\
T2 : & \quad \lambda_1 = \frac{1}{10\sqrt{2}} \quad \lambda_2 = \frac{1}{10\sqrt{2}} \quad \sqrt{\lambda_1^2 + \lambda_2^2} = \frac{1}{10} \quad \frac{\lambda_2}{\lambda_1} = 1 \\
T3 : & \quad \lambda_1 = \frac{2}{\sqrt{10}} \quad \lambda_2 = \frac{2}{3\sqrt{10}} \quad \sqrt{\lambda_1^2 + \lambda_2^2} = \frac{2}{3} \quad \frac{\lambda_2}{\lambda_1} = \frac{1}{3}
\end{align*}
\]

Estimate \( \gamma_1 \) och \( \gamma_2 \) from the filter plots:

Filter 1 : \( \gamma_1 = 0.5 \quad \gamma_2 = 0.5 \)
Filter 2 : \( \gamma_1 = 0 \quad \gamma_2 = 0 \)
Filter 3 : \( \gamma_1 = 1 \quad \gamma_2 = 0 \)

The control tensor \( \mathbf{C} = \gamma_1 \mathbf{e}_1 \mathbf{e}_1^T + \gamma_2 \mathbf{e}_2 \mathbf{e}_2^T = \gamma_1 \mathbf{e}_1 \mathbf{e}_1^T + \gamma_1 \mu_2 \mathbf{e}_2 \mathbf{e}_2^T \)

The \( m \)-function should pass trough \((0.1,0), (0.5,0.5) \) and \((0.67,1)\).

The \( \mu \)-function should pass trough the points \((0.33,0), (0.5,1)\).