Answers to the exam in  
Medical image analysis - TBMI02  
2011-04-26

The answers presented below may be more exhaustive compared to what is normally required at the exam.

Part 1

1. We first write the equation for the affine field on the form \( \mathbf{v}(x) = \mathbf{B}(x) \mathbf{p} \),

\[
\mathbf{v}(x) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} p_3 & p_4 \\ p_5 & p_6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & y \end{bmatrix} \mathbf{B} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \]

(1)

where \( \mathbf{p} = [p_1 \ p_2 \ p_3 \ p_4 \ p_5 \ p_6]^T \). A two dimensional scaling and rotation matrix can be written as

\[
\mathbf{R} = s \mathbf{I} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}
\]

(2)

where \( \mathbf{I} \) is the identity matrix. We need two parameters to describe the rotation and the scaling, one for \( \theta \) and one for \( s \). The translations are, as before, described by two parameters.

\[
\mathbf{v}(x) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} p_3 & p_4 \\ -p_4 & p_3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & y & -x \end{bmatrix} \mathbf{B} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \]

(3)

where \( \mathbf{p} = [p_1 \ p_2 \ p_3 \ p_4]^T \). Isotropic scaling can for example be achieved by using \( p_4 = 0, p_3 = 2 \).

2. Apply a magnetic gradient field in a direction perpendicular to the desired image plane. Transmit a radio frequency pulse with a frequency determined by the Larmor frequency of the spins in the slice that you want to excite. Only spins where the two frequencies are close enough will be excited. Thus, the bandwidth of the radio frequency pulse transmitted and the strength of the magnetic gradient will determine how thick the excited slice will be.

3. True False

1. The \( \gamma_1 \)-image has low magnitude in areas that just contain noise. \( \boxed{x} \) \( \boxed{\ } \)

2. The \( \mu_2 \) mapping is independent of \( \|\mathbf{T}\| \). \( \boxed{x} \) \( \boxed{\ } \)

3. The lowpass filter is used only when \( \|\mathbf{C}\| = 0 \). \( \boxed{\ } \) \( \boxed{x} \)

4. If the SNR in the image is increased the noise threshold in the \( m \)-function should be decreased. \( \boxed{x} \) \( \boxed{\ } \)
4. The easiest way might be to use region growing. Start in the point given by the mouse pointer and study its neighbours. If they are similar enough compared to the first point they are also allowed to be a part of the object. Then study their neighbours and determine if they are similar enough to their neighbours. Continue like this and set the border of the object where the neighbouring pixels are too different. This method is easy and fast and it is easy to understand the meaning of the parameters. It is though rather sensitive to noise (which the Magic Wand in Photoshop also is). This problem could be solved by some noise reduction algorithm, for example a simple lowpass filtering.

5. The Nyquist Frequency \( f = \pi \) corresponds to a wavelength of two pixels \( (f = 2\pi/\lambda) \) which we interpret as an object size of one pixel. An object of size 1.5 and 6 pixels corresponds consequently to a filter with the upper and lower 6dB frequencies \( f_1 = 2\pi/3 \) och \( f_2 = 2\pi/12 \). The relative bandwidth of the filter \( \beta = \ln(f_1/f_2)/\ln(2) = 2 \) octaves and the center frequency is computed as the geometric mean value \( f_0 = \sqrt{f_1f_2} = \pi/3 \).

6. The frequency weight should reflect the spectrum of the image. Frequencies that occur more frequently are more important. For most natural images the spectrum is proportional to \(|u|^{-\alpha}\) where \(\alpha \in [1, 2]\). If the image contains a lot of noise this will affect the spectrum. The noise is broad banded and is most simple to model by adding an offset to the frequency weight function.

7. \[
\min \epsilon^2 = \sum_{x=-1}^{1} [w(x)(\alpha + \beta x - 4x)]^2 \quad w(x) = [2, 1, 1]
\]

\[
\epsilon^2 = 2^2 (\alpha - \beta - 0.25)^2 + 1^2 (\alpha - 1)^2 + 1^2 (\alpha + \beta - 4)^2
\]

Compute \( \frac{d\epsilon^2}{d\alpha} = 0 \) and \( \frac{d\epsilon^2}{d\beta} = 0 \) and solve \( \alpha = 1.86 \) och \( \beta = 1.71 \).

weighted error = \( \sqrt{\sum_{x=0}^{2}[w(x)(\alpha + \beta x - 4x)]^2} / \sum_{x=0}^{2}w(x)^2 = 0.40 \)
8. Equations (1) and (2) give the relative signal amplitudes for two different types of tissue as a function of repetition time. Which repetition time, $T_R$, gives the highest signal difference (contrast) between the two tissue types?

$$S_{\text{tissue1}}(T_R) = 1 - e^{-2T_R} \quad (4)$$
$$S_{\text{tissue2}}(T_R) = 1 - e^{-T_R} \quad (5)$$

$T_R = \ln(2)$

Solution:

The signal difference is:

$$S_{\text{diff}}(T_R) = S_{\text{tissue1}}(T_R) - S_{\text{tissue2}}(T_R) = e^{-T_R} - e^{-2T_R} \quad (6)$$

The derivative of $S_{\text{diff}}(T_R)$ with respect to $T_R$ is:

$$S'_{\text{diff}}(T_R) = 2e^{-2T_R} - e^{-T_R} \quad (7)$$

Maximum difference implies $S'_{\text{diff}}(T_R) = 0$, i.e.

$$2e^{-2T_R} = e^{-T_R} \quad (8)$$

Take the logarithm of both sides

$$\ln(2) - 2T_R = -T_R \quad (9)$$

Which gives the answer: $T_R = \ln(2)$ (max difference $= 0.25$)

9. An image of an object in an MR scanner has been acquired. The sampling in k-space is just right for imaging the object at the standard resolution without any aliasing effects. Why and how should the sampling in k-space be changed if

a) we want an image of the object at a resolution that is twice as high?

**ANSWER:** Sample k-space out to twice the previous distance at the same sample density as before. (This will in 3D give eight times as many k-space samples.)

b) we want to scan an object that is twice the size with the standard resolution?

**ANSWER:** Sample k-space out to the same frequency as before but double the sample density. (This will in 3D also give eight times as many k-space samples.)

10. If the Fourier transform of $f(x, y)$ is given by $F(u, v)$ the Fourier transform of $f(x, \beta y)$ is given by:

$$|\beta|^{-1} F(u, \beta^{-1}v).$$
11. An image is described by:

\[ I(x) = \pi + \sin(\hat{n}^T x) \]  

(10)

A perfect local structure estimation gives the same tensor, \( T \), in all image neighborhood. \( T \) is given by:

\[ T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]  

(11)

Find \( \hat{n} \), the normalized vector determining the orientation of the sine wave.

Answer

\[ \hat{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

Solution: We know that \( \hat{n} \) is an eigenvector of \( T \), i.e \( T\hat{n} = \lambda \hat{n} \). Letting \( \hat{n} = (x, y)^T \) we get:

\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}
\]

yielding:

\[
\begin{pmatrix} x + y \\ x + y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}
\]

We can directly see that \( x = y \) and since \( \hat{n} \) is a normalized vector we have found the answer.

12. The physiological mechanism that makes functional magnetic resonance imaging (fMRI) possible is called the BOLD effect. What is this abbreviation short for and what is the relation to neural activity?

**ANSWER:** BOLD is short for 'Blood Oxygen Level Dependent'. A relatively higher concentration of oxygenated blood is found close to active neurons.
Part 2

13. a) In each iteration of the algorithm interpolation is used to get the image at the new location. Interpolation has a similar effect as lowpass filtering and a repeated lowpass filtering will lead to a result with a very low quality. If $p$ instead is updated in each iteration, the interpolation will always be performed from the original image and the lowpass effect will thereby only occur once.

b) The motion can locally be described as a movement (i.e. there is no intensity difference between the images).

$$I_2(x + v(x)) \approx I_1(x)$$

II The intensity in the image can locally be described as a slanted plane (i.e. as a first order Taylor expansion).

$$I(x + v(x)) \approx I(x) + v(x)^T \nabla I(x)$$

c) The two assumptions above are better suited for the local phase than for the image intensity.

14. a) See lecture notes.

b) The problem is that the snake will get stuck in convex hulls if only the arc length of the snake is minimized (the snake must sometimes become longer to adapt to the object, remember the balloon hand in the laboration). An easy solution is to always point the internal forces towards the object.

15. a) We have $T = \lambda_1 \hat{e}_1 \hat{e}_1^T + \lambda_2 \hat{e}_2 \hat{e}_2^T$. The identity matrix $I$, which we here interpret as an isotropic tensor with $\lambda_1 = \lambda_2 = 1$, can be expressed in many ways. We choose to use the eigensystem of $T$

$$I = \hat{e}_1 \hat{e}_1^T + \hat{e}_2 \hat{e}_2^T$$

The local tensor can now be expressed as:

$$T = (\lambda_1 - \lambda_2) \hat{e}_1 \hat{e}_1^T + \lambda_2 I$$

$$T - \lambda_2 I = (\lambda_1 - \lambda_2) \hat{e}_1 \hat{e}_1^T$$

Since we know that $\|\hat{e}_1 \hat{e}_1^T\| = 1$ the most robust computation of $\hat{e}_1 \hat{e}_1^T$ is to normalize:

$$\hat{e}_1 \hat{e}_1^T = (T - \lambda_2 I)^{-1} (T - \lambda_2 I)$$

and finally $\hat{e}_2 \hat{e}_2^T = I - \hat{e}_1 \hat{e}_1^T$.

b) $T$ is a second order local structure tensor in 3D.

$$T = \lambda_1 \hat{e}_1 \hat{e}_1^T + \lambda_2 \hat{e}_2 \hat{e}_2^T + \lambda_3 \hat{e}_3 \hat{e}_3^T \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$$

Following the hint we compute:

$$T_1 = (\lambda_2 - \lambda_1) \hat{e}_2 \hat{e}_2^T + (\lambda_3 - \lambda_1) \hat{e}_3 \hat{e}_3^T$$

$$T_2 = (\lambda_1 - \lambda_2) \hat{e}_1 \hat{e}_1^T + (\lambda_3 - \lambda_2) \hat{e}_3 \hat{e}_3^T$$

$$T_3 = (\lambda_1 - \lambda_3) \hat{e}_1 \hat{e}_1^T + (\lambda_2 - \lambda_3) \hat{e}_2 \hat{e}_2^T$$
\[ \mathbf{T}_{12} = \mathbf{T}_1 \mathbf{T}_2 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1) \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2^T + \\
+ (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1) \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_3^T + \\
+ (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2) \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3^T + \\
+ (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3^T = 0 \]

We now compute \( \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3^T \) by normalizing \( \mathbf{T}_{12} \)

\[ \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3^T = \| \mathbf{T}_{12} \|^{-1} \mathbf{T}_{12} \]

Then Compute \( \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2^T = \| \mathbf{T}_{13} \|^{-1} \mathbf{T}_{13} \) in the same manner. Finally \( \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1^T = \mathbf{I} - \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2^T - \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3^T \).

16. From the figure we get

\[ \mathbf{T}_1 = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

The local tensor and the resulting control tensor.

Prior to the \( \mu \)-map we have the relation \( \lambda_2/\lambda_1 = 0.33 \). From the figure we get \( \mu(0.33) = 0.1 \) which implies the \( \lambda_2/\lambda_1 = 0.1 \) for the control tensor.

\[ \mathbf{C} = \gamma_1 \mathbf{e}_1 \mathbf{e}_1^T + \gamma_1 \mu_2 \mathbf{e}_2 \mathbf{e}_2^T = \frac{1}{40} \begin{pmatrix} 11 & -9 \\ -9 & 11 \end{pmatrix} \]

for \( \gamma_1 = 0.5 \).